Asymptotic Collusion-proofness of Voting Rules and Incomplete Information Setting

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Abstract

Classical results in voting theory show that strategic manipulation by voters is inevitable if a voting rule has to simultaneously satisfy certain desirable properties. Motivated by this, we study the relevant question of how often a voting rule is manipulable. It is well known that elections with a large number of voters are rarely manipulable under impartial culture (IC) assumption. However, the manipulability of voting rules when the number of candidates is large has hardly been addressed in the literature and our work focuses on this problem. First, we propose two properties (1) asymptotic strategy-proofness and (2) asymptotic collusion-proofness, with respect to new voters, which makes the two notions more relevant from the perspective of computational problem of manipulation. In addition to IC, we explore a new culture of society where all score vectors of the candidates are equally likely. This new notion has its motivation in computational social choice and we call it impartial scores culture (ISC) assumption. We study asymptotic strategy-proofness and asymptotic collusion-proofness for plurality, veto, k-approval, and Borda voting rules under IC as well as ISC assumptions. Specifically, we prove bounds for the fraction of manipulable profiles when the number of candidates is large. Our results show that the size of the coalition and the tie-breaking rule play a crucial role in determining whether or not a voting rule satisfies the above two properties.

Gibbard Satterthwaite theorem applies for the complete information setting where manipulators know the votes of the truthful voters. But a more realistic scenario is to have the manipulators who only have a non-deterministic belief over the votes of the truthful voters. This setting is called incomplete information setting. We study manipulability of different voting rules in the incomplete information setting and provide characterization of the belief under which the voting rules will be strategy-proof.
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Chapter 1

Introduction

In many real life situations including multi-agent systems, agents often need to agree upon a common decision although they may have different preferences over the possible alternatives. The central problem in this scenario is preference aggregation, that is, aggregating different preferences and agreeing on a joint plan. Voting is a special type of preference aggregation method that has been applied in many current problems, for example, collaborative filtering [17], planning among multiple automated agents [6], etc. A wide variety of voting rules have been proposed in the literature. A voting rule in conjunction with the population of voters and candidates is commonly referred to as an election.

1.1 Manipulation of voting rules

A fundamental problem of voting rules is the strategic manipulation by voters - sometimes voters are better off by voting non-truthfully. Informally, a voter is said to manipulate an election if she does not report her true preference. Depending upon the voting rule, and the votes what other voters are casting, a voter may be better off by reporting something which is not her true preference about the possible alternatives. For example, consider the plurality election below. In plurality election voters vote for their best candidate and the candidate receiving most votes wins the election. Consider a plurality election with three candidates say $a$, $b$, and $c$. Suppose a voter knows that from the votes other than her, $a$ receives 40% of the total votes, $b$ also receives 40% of the total votes, and $c$ gets rest of the votes. If the voter’s actual preference is $c \succ a \succ b$, then she gets better outcome by casting her vote in favor of $a$ instead of $c$. Hence the voter is better off by non-truthful voting. This phenomenon is called manipulation.
1.2 Gibbard-Satterthwaite Theorem

Manipulation is clearly undesirable at least for the reason that a candidate may end up winning the election who would not be the winner had all voters voted truthfully. The immediate game theoretic question that crops out in this context is does there exist voting rules for which manipulation is impossible. That is a voter can never be better off by voting non-truthfully. Certainly the answer is yes. Consider a voting rule which elects a fixed candidate irrespective of the votes of the voters, that is the voting rule is a constant function. But this voting rule is clearly not interesting. This voting rule at times will not even select a candidate even when that candidate is the most preferred candidate of all the voters which seems to have a minimum requirement we want our voting rule to respect. This property is called *unanimity*. Hence an interesting question is whether there exists an unanimous voting rule which is non-manipulable. This answer also turns out to be yes. Consider the following voting rule - the voting rule selects the most preferred candidate of a predetermined fixed voter say $v_d$. This voting rule satisfies the unanimity property discussed above. However the problem of this kind of voting rules is that it does not meet the fundamental requirement of voting rules, that it is a preference aggregation method - this rule completely ignores the votes of all the voters except a particular voter. This type of voting rules are called *dictatorial* voting rules and the voter whose most preferred candidate is the winner is the *dictator*. Hence we want to avoid these kind of voting rules in our quest to find non-manipulable voting rules. But the following celebrated negative result in the context of voting theory preclude the existence of such a voting rule. The Gibbard Satterthwaite (abbreviated as G-S) theorem [9, 21] shows that every unanimous and non-dictatorial voting rule with at least three candidates is manipulable. The G-S theorem is applicable to the elections with at least three candidates. In elections with two candidates, May’s theorem [14] says that plurality is the only election in this setting which simultaneously satisfies all the three desirable properties namely unanimity, non-dictatorship, and non-manipulability.

1.3 Preliminaries and Relevant Work

Following the impossibility result of Gibbard and Satterthwaite, there are multiple attempts to circumvent the result by tweaking some of the assumptions of the theorem. One way out that Gibbard himself immediately pursued is to consider randomized voting rules. Classically voting rules outputs a candidate (winner) for each voting profile. In randomized voting rules, the output is a probability distribution over the candidates. Gibbard [10] showed that the non-manipulable randomized voting rules are only the voting rules which is a probability mixture
of unilateral and duple voting rules. A voting rule is unilateral if only one voter affects the outcome. Note that unilateral voting rules are not same as dictatorial rules although it includes all the dictatorial rules. Random dictatorship is an example of a voting rule which is unilateral but not dictatorship. In random dictatorship voting rule, a dictator is chosen for each profile and her most preferred candidate is the winner. A randomized voting rule is called duple if only two candidates can win in any voting profile. Hence both unilateral and duple voting rules are uninteresting since both rules fail severely to aggregate preferences - a unilateral voting rule ignores all voters except one and a duple voting rule ignores all candidates except two predetermined candidates.

1.4 Domain Restriction

Economists tried to bypass the G-S theorem by restricting the domain of the voting rule. One domain popular among political scientists and political economists is the single peaked preference domains defined below.

**Definition 1.1** (Single Peaked Preference [13])

A preference \( \succ \) is single peaked with respect to an order \( > \) on a set \( X \) if there is an alternative \( x \in X \) with the property that \( > \) is increasing with respect to \( > \) on \( \{ y \in X : x > y \} \) and decreasing with respect to \( > \) on \( \{ y \in S : y > x \} \). That is,

\[
x > z > y \Rightarrow z \succ y
\]
\[
y > z > x \Rightarrow z \succ y
\]

A domain is called single peaked if the preferences of all voters’ are single peaked. It is well known that for single picked preference domain, median voting rule simultaneously satisfies unanimity, non-dictatorship, and non-manipulability [13]. However the drawback of domain restriction is that, the designer frequently will not be sure whether the preferences of the voters fall in the restricted domain or not. If the preferences happen not to fall in the domain then the results do not hold.

1.5 Asymptotic Strategy-proofness

In this context, a natural research direction is to look for voting rules which are rarely manipulable. Note that a voting rule being manipulable only means that there exists at least one preference profile where manipulation is possible. In general, in an election with \( n \) voters and \( m \) candidates, all of \( m!^n \) voting profiles may not be manipulable. For example consider
a plurality election with three candidates say $a$, $b$, and $c$ where both $a$ and $b$ get 10% of total votes and $c$ gets the rest of the votes. This preference profile is not manipulable since a new voter cannot change the outcome by casting any preference. Hence she does not have any any incentive to lie (In game theory, a standard assumption is that if a player does not have any incentive of lying then she will behave truthfully). Another example of preference profile where manipulation is not possible is the plurality election where all the candidates get equal vote. On the other hand, a plurality election where $a$ and $b$ each get 50% of the votes and $c$ does not receive any vote, is certainly manipulable. Hence we look for voting rules for which most of the profiles are non-manipulable. Pattanaik [16] conjectured that “the possibility of strategic voting by single individuals will be smaller the greater the number of the individuals.” Pattanaik’s above conjecture has been subsequently verified. Slinko [22] showed that for some common voting rules, under impartial culture (IC) assumption, the probability of drawing a manipulable profile at random goes to zero as the number of voters increases. Impartial culture assumption says that the voters’ preferences are independent and uniformly distributed among all possible linear orders of the candidates. The formal definition is as follows.

**Definition 1.2 (Impartial Culture Assumption)**

For any two $n$-voter preference profile $(\succ_1, \ldots, \succ_n)$ and $(\succ'_1, \ldots, \succ'_n)$,

$$\text{Prob}(\succ_1, \ldots, \succ_n) = \text{Prob}(\succ'_1, \ldots, \succ'_n)$$

where $\text{Prob}(\succ_1, \ldots, \succ_n)$ and $\text{Prob}(\succ'_1, \ldots, \succ'_n)$ denotes the probability that the actual profile of the voters is $(\succ_1, \ldots, \succ_n)$ and $(\succ'_1, \ldots, \succ'_n)$ respectively.

Slinko also showed similar results under impartial anonymous culture assumption which says the number of times any preference occurs in a profile is uniformly distributed. Formally it is defined as follows.

**Definition 1.3 (Impartial Anonymous Culture Assumption)**

Let $x_1, \ldots, x_M$ be the $M = m!$ preferences with $m$ candidates. Then for any two $(k_1, \ldots, k_M) \in \mathbb{N}^M$ such that $\sum_{i=1}^M k_i = n$ and $(k'_1, \ldots, k'_M) \in \mathbb{N}^M$ such that $\sum_{i=1}^M k'_i = n$,

$$\text{Prob}(k_1, \ldots, k_M) = \text{Prob}(k'_1, \ldots, k'_M)$$

where $\text{Prob}(k_1, \ldots, k_M)$ and $\text{Prob}(k'_1, \ldots, k'_M)$ denotes the probability that the preference $x_i$ occurs $k_i$ number of times for all $i$, and $x_i$ occurs $k'_i$ number of times for all $i$ respectively.

Slinko called such voting rules *asymptotically strategy-proof*. However the above result does not directly connect the computational problem of manipulation since it defines manipulable
profiles through the existence of manipulation by current voters in the profile. Also he studied
asymptotic strategy-proofness by tending the total number of voters towards infinity keeping
the total number candidates fixed. We studied the asymptotic behavior of manipulation with
respect to the total number of candidates. We argue that this scenario is more interesting
from the computational voting theory point of view. We, in this work, propose a definition of
asymptotic strategy-proofness with respect to new voters which makes it consistent with the
computational problem of manipulation which is discussed below.

1.6 Coalitional Manipulation

Bartholdi and coauthors [2, 1] asked the following question: what if finding a manipulating
preference is computationally so hard that the agents will not be able to find it? This led to the
formulation and study of the computational problem of coalitional manipulation (CM) which
is defined as follows.

Definition 1.4 (Coalitional Manipulation)

Given

- a voting rule \( r \).
- a set of candidates \( C \). \( |C| = m \).
- a candidate \( x \in C \).
- a voting profile of \( n \) non-manipulators.
- a set of manipulators \( M \). \( |M| = c \).

Find

- Does there exist \( P^M \) such that \( |P^M| = c \) and \( r(P^{NM}, P^M) = x \)?

Bartholdi and Orlin showed that the CM problem is \( \text{NP} \)-complete for Single Transferable Vote
(STV) voting rule even when \( c = 1 \). The STV voting rule is as follows. Suppose we have \( m \)
number of candidates. Then STV proceeds in \( (m - 1) \) iterations. In each iteration, plurality
rule is used and the candidate with least number of plurality score drops out and its votes
gets transferred to the next candidate in the preferences. If there are more than one candidate
with least plurality score then a tie breaking rule is used. In plurality vote, the candidate at
top in a preference gets a score of one and rest gets zero score from that preference. The CM

\footnote{A \( n \)-voters' voting profile is \( (\succ_1, \ldots, \succ_n) \), where each \( \succ_i \) is a linear order over \( C \).}
problem for many common voting rules including copeland, Borda, maximin, ranked pair etc. has been shown to be \( \text{NP} \)-complete for \( c > 1 \) [7, 3, 26]. This computational intractability seems to provide a kind of barrier against manipulation for many common voting rules.

### 1.7 Complexity - a weak barrier

However, since intractability only provides a worst case guarantee, how much protection computational hardness provides in actual practice, is questionable - what if the computationally hard instances are only pathological situations and rarely occur in practice. This has been partially answered by Procaccia and Rosenschein [20] by showing average case easiness of the manipulation problem assuming junta distributions over the voters’ preferences. Conitzer and Xia [25] showed that starting with a uniformly random voting profile and then randomly resetting the ranking of one of the voters yields a manipulation pair with probability \( \Omega(\frac{1}{n}) \). But the properties their social choice function must satisfy is quite stringent. Although they showed that many common voting rules satisfy their assumptions but many do not. Hence their result loses generality. Dobzinski and Procaccia [5] prove analogous result of Conitzer’s. Their assumption on voting rule is very weak and thus most voting rules satisfy their assumption. However their result holds only for two voters case, that is for \( n=2 \) only which dampens its universality.

Friedgut, Kalai, and Nisan [8] showed that if \( f \) is a neutral voting rule which is \( \epsilon \)-dictatorial and total number of candidates is 3, then \( \exists C > 0 \) such that \( \text{Prob}\{(X,Y) \text{ is a manipulable pair of } f\} \geq \frac{C \epsilon^2}{n} \). Hence for sufficiently non-dictatorial voting rules and for 3 candidates, random manipulation succeeds with non-negligible probability. A voting rule is called \( \epsilon \)-dictatorial if it differs from all dictatorship rules at at least \( \epsilon \) fraction of the domain. The formal definition is as follows.

**Definition 1.5 (\( \epsilon \)-dictatorial)**

Consider an election setting with \( m \) number of candidates and \( n \) number of voters. A voting rule \( r \) is called \( \epsilon \)-dictatorial if for all dictatorial voting rule \( r_d \),

\[
|\{(\succ_1, \ldots, \succ_n) : r(\succ_1, \ldots, \succ_n) \neq r_d(\succ_1, \ldots, \succ_n)\}| \geq \epsilon m! n.
\]

Isaksson, Kindler, and Mossel [11] proved the following result: Let \( f \) be a neutral voting rule and \( f \) is \( \epsilon \)-dictatorial. Then \( \text{Prob}\{(\succ, \succ') \text{ is a manipulation pair of } f\} \geq \frac{\epsilon^2}{10^3 n^3 m^2} \). A pair of profiles \( \succ \) and \( \succ' \) forms a manipulation pair if they differ only at some \( i^{th} \) player’s preference and manipulation is possible by the \( i^{th} \) player. Formally,

**Definition 1.6 (Manipulation Pair)**

Two preference profiles \((\succ_i, \succ_{-i})\) and \((\succ'_i, \succ'_{-i})\) is called a manipulation pair with respect to a
voting rule \( r \) if,

\[ r(\succ_i, \succ_{-i}) \succ_i r(\succ_i, \succ_{-i}) \]

What this result implies is that for sufficiently non-dictatorial voting rules, random manipulation has non-negligible probability of success. This immediately gives a randomized manipulation algorithm for sufficiently non-dictatorial voting rules which succeeds with non-negligible probability on average. This also gives a quantitative version of the G-S theorem. Researchers also looked for experimental evidences to find the worthiness of complexity barrier against manipulation. Toby Walsh [24] empirically showed that STV voting rule is almost always easy to manipulate for a number of distributions over votes including uniform and real world elections. Note that STV is the first among the widely used voting rules that has been shown to be intractable to manipulate. STV is also among the very few voting rules which are intractable to manipulate even by one manipulator. In a nutshell, STV is believed to be one of the hardest to manipulate voting rules. Davies, Katsirelos, Narodytska, and Walsh [4] provide heuristic methods namely largest fit, average fit, and reverse using ideas from bin packing and multi-processor scheduling for manipulating Borda voting rule. Their experimental results suggest that average fit successfully manipulates 99% of the time although they showed that the CM problem for Borda is \( \text{NP} \)-complete even with two manipulators. However to the best of our knowledge, the following reverse question still remains unanswered - given the inputs where the CM problem is actually hard, how much severe is the problem of manipulation on those inputs? This question is justified as there is no point in trying to stop manipulation at profiles which are game theoretically strategy-proof. We address this question in this work.

1.8 Motivation

It is known that the CM problem can be solved in \( O \left( \left( c + m! \right)^m \cdot \text{poly} \left( m, n, c \right) \right) \) time\(^1\) for any anonymous and efficient\(^2\) voting rule. The computational complexity arises from the fact that there are \( O \left( \left( c + m! \right)^m \right) \) different ways to distribute the \( c \) votes of manipulators among \( m! \) possible preferences. Hence all the hard instances of the CM problem are elections with a large number of candidates. This motivates us to study the severity of manipulation in elections with a large number of candidates. Asymptotic strategy-proofness has been classically studied keeping the number of candidates fixed and varying only the number of voters. Hence the classical asymptotic strategy-proofness fails to connect itself with computational problem of manipulation since for constant number of candidates, complexity barrier does not exist. Also

\(^1\)poly\((m, n, c)\) denotes the set of all polynomial functions over \( m, n \), and \( c \).

\(^2\)A voting rule is efficient if winner determination is \( \text{poly}(m, n) \) time computable.
Nitzan [15] empirically showed that the problem of manipulation is severe only in societies with small number of voters. Hence it is the elections with small number of voters which we should possibly target for preventing manipulation. However small elections with only a few candidates are always easily manipulable - manipulators can just try all possible linear orders over the candidates. This leaves only one case open - elections with fixed number of voters but large number of candidates.

In another direction, we study manipulation in the incomplete information setting. Gibbard-Satterwaite theorem assume complete information setting. In this setting, manipulators know the votes of the non-manipulators. Although this setting is far from being realistic, this has been the setting studied extensively in the literature. The motivation behind this is to give manipulators as much power as possible. But with vast number of results showing inevitability of manipulation in this setting, it might be the time to look at the more realistic setting - the incomplete information setting. In this setting, manipulators only have a probabilistic belief about the votes of the manipulators. We provide characterization results for manipulability in this setting.

1.9 Contributions

Besides active theoretical interest, there are many practical scenarios as well where the number of candidates could be large, for example, meta search engines, employee selection by a committee, etc. In this work, we study the manipulability of voting rules when the number of voters is fixed and the number of candidates increases to infinity. The specific contributions of this work are as follows.

- We define the notions of asymptotic strategy-proofness and asymptotic collusion-proofness with respect to new voters (definition 2.5) which makes the notions more relevant from the perspective of computational problem of manipulation. We show that the existing results on the manipulability for plurality, veto, and $k$-approval voting rules continue to hold under the proposed definition of asymptotic strategy-proofness.

- We explore a new culture of society where all score vectors of the candidates are equally likely. This new notion has its motivation in computational social choice. We call this assumption impartial scores culture (ISC) assumption.

- We provide bounds for the fraction of manipulable profiles for voting rules when the number of candidates is large. These bounds immediately tell us whether or not the given voting rule is asymptotically collusion-proof. We prove asymptotic results for plurality,
veto, and \( k \)-approval voting rules under both IC and ISC assumptions. We prove Borda rule is not asymptotic strategy-proof on the number of candidates under the IC assumption but the problem is still unresolved under the ISC assumption. Our results show that the size of the coalition and the tie-breaking rule being used play a crucial role in determining whether or not a voting rule is asymptotically strategy-proof.

- We explore manipulation in the incomplete information setting and prove characterizations of manipulability in that scenario.

<table>
<thead>
<tr>
<th>Cases</th>
<th>Plurality for(^a)</th>
<th>Plurality against(^b)</th>
<th>Veto</th>
<th>( k )-Approval, ( k &gt; 1 )</th>
<th>Borda</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c = 1 ), IC</td>
<td>(\checkmark)</td>
<td>(\times)</td>
<td>(\times)</td>
<td>( k = O(1) : \times )</td>
<td>(\times)</td>
</tr>
<tr>
<td>( c = 1 ), ISC</td>
<td>(\checkmark)</td>
<td>(\times)</td>
<td>(\times)</td>
<td>( k = O(1) : \times )</td>
<td>?</td>
</tr>
<tr>
<td>( c &gt; 1 ), IC</td>
<td>(\times)</td>
<td>(\times)</td>
<td>(\times)</td>
<td>( k = O(1) : \times )</td>
<td>(\times)</td>
</tr>
<tr>
<td>( c &gt; 1 ), ISC</td>
<td>(\times)</td>
<td>(\times)</td>
<td>(\times)</td>
<td>( k = O(1) : \times )</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 1.1: Asymptotic collusion-proofness results on candidates

\(^a\)Plurality Rule with lexicographic tie breaking rule for the manipulators
\(^b\)Plurality Rule with lexicographic tie breaking rule against the manipulators

<table>
<thead>
<tr>
<th>Cases</th>
<th>Plurality</th>
<th>Veto</th>
<th>( k )-Approval, ( k &gt; 1 )</th>
<th>Borda</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c = o(n) ), ISC</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>( c = o(n) ), IC</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 1.2: Asymptotic collusion-proofness results on voters
Chapter 2

Notation and Definitions

Let \( V = \{v_1, v_2, \ldots, v_n\} \) be the set of voters and \( C = \{c_1, c_2, \ldots, c_m\} \) the set of candidates. Each voter \( v_i \) has a preference \( \succ_i \) over the candidates which is a linear order over \( C \). The set \( L(C) \) denotes the set of all linear orders over \( C \). A map \( r_c : \powerset{\mathbb{N}^+ \times L(C)}^n \rightarrow 2^C \setminus \emptyset \) is called a voting correspondence\(^1\). A map \( t : 2^C \setminus \emptyset \rightarrow C \) such that \( t(A) \in A, \forall A \in 2^C \setminus \emptyset \), is called a tie breaking rule. Commonly used tie breaking rules are lexicographic tie breaking rule where ties are broken in accordance with a \( \succ_t \in L(C) \). We, in this work, study two types of tie breaking rules - lexicographic tie breaking rule where ties are broken in favor of the manipulators and against the manipulators respectively. A voting rule is \( r = t \circ r_c \). From here on, if not mentioned \( n \) denotes the number of voters and \( m \) denotes the number of candidates.

For example, \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m \) with \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \) naturally defines a voting rule - a candidate gets score \( \alpha_i \) from a vote if it is placed at the \( i^{th} \) position and winner is the candidate with maximum score. This voting rule is called a positional scoring rule based voting rule with score vector \( \alpha \). For \( \alpha = (m - 1, m - 2, \ldots, 1, 0) \), we get the Borda voting rule. With \( \alpha_i = 1 \ \forall i \leq k \) and 0 else, the voting rule we get is known as \( k \)-Approval. Plurality is \( 1 \)-Approval and veto is \( (m - 1) \)-Approval. Next we define some important axioms that we expect a voting rule to satisfy.

2.1 Axiomatic Properties of Voting Rules

**Definition 2.1 (Anonymity)** A voting rule is called anonymous if the names of the voters are

\(^1\)By \( \uplus \), we denote disjoint union. \( \mathbb{N}^+ := \{1, 2, 3, \ldots\} \). \( 2^C \) denotes the power set of \( C \). \( \emptyset \) denotes the empty set.

\(^2\)\( \circ \) denotes composition of mappings.
irrelevant. Formally, a voting rule \( r \) is anonymous if
\[
\forall n \in \mathbb{N}^+, \forall (\succ_1, \ldots, \succ_n) \in \mathcal{L}(\mathcal{C})^n, \forall \sigma \in S_n \quad r(\succ_1, \ldots, \succ_n) = r(\succ_{\sigma(1)}, \ldots, \succ_{\sigma(n)})
\]

**Definition 2.2 (Neutrality)** A voting rule is called neutral if the names of the candidates are immaterial. That is,
\[
\forall n \in \mathbb{N}^+, \forall (\succ_1, \ldots, \succ_n) \in \mathcal{L}(\mathcal{C})^n, \forall \sigma \in S_m \\
r(\sigma \circ \succ_1, \ldots, \sigma \circ \succ_n) = \sigma \circ r(\succ_1, \ldots, \succ_n)
\]

**Definition 2.3 (Strategy-proofness)** A voting rule is called strategy-proof if voters are not worse off by reporting true preferences than reporting any other preference. That is,
\[
\forall n \in \mathbb{N}^+, \forall \succ^n \in \mathcal{L}(\mathcal{C})^n, \forall \succ, \succ' \in \mathcal{L}(\mathcal{C}), \\
r(\succ^n, \succ) \succ r(\succ^n, \succ')
\]

A non-strategy-proof voting rule is called manipulable. The Gibbard and Satterthwaite Theorem says that, any unanimous, and non-dictatorial voting rule with at least three candidates is necessarily manipulable.

### 2.2 Collusion-proof Profiles

Given a voting rule \( r \), a set of candidates \( \mathcal{C} \), \( n \) number of truthful voters, and a coalition size \( c \), we define \( c \)-collusion-proof voting profiles as follows.

**Definition 2.4 (\( c \)-Collusion-proof Voting Profile)** A voting profile \( \succ^n \in \mathcal{L}(\mathcal{C})^n \) is called \( c \)-collusion-proof if \( \forall \succ_1^c, \succ_2^c \in \mathcal{L}(\mathcal{C})^c \),
\[
r(\succ^n, \succ_1^c) \succ_1^c r(\succ^n, \succ_2^c)^2
\]

For \( c = 1 \), these profiles are called strategy-proof voting profiles. The definition is as follows.

**Definition 2.5 (Strategy-proof Voting Profile)** A voting profile \( \succ^n \in \mathcal{L}(\mathcal{C})^n \) is called strategy-proof if \( \forall \succ_1, \succ_2 \in \mathcal{L}(\mathcal{C}) \),
\[
r(\succ^n, \succ_1) \succ_1 r(\succ^n, \succ_2)
\]

\(^1\) \( S_n \) denotes the set of all permutations of the set \( \{1, 2, \ldots, n\} \).

\(^2\) We say \( a(\succ_1, \ldots, \succ_n) b \) if \( a \succ_1 b \forall i = 1, 2, \ldots, n \).
Given a voting rule \( r \), we denote the set of all \( c \)-collusion-proof voting profiles by \( T^c_r(C) \). The set of all strategy-proof voting profiles is denoted by \( T_r(C) \). We will drop the subscript \( r \) whenever the context is clear. Notice that manipulation by \( c \) number of new voters is game theoretically not possible in the above defined collusion-proof profiles. We define the above notions with respect to new voters which directly implies that the hardness of the CM problem at collusion-proof instances is of no value. Previously, Slinko [22] defined strategy-proof voting profiles with respect to the existing voters. Obviously, our definition is more aligned with the formulation of the CM problem and hence the results give direct implications of the usefulness the CM problem being hard at a particular instance.

If \( r \) is a score based voting rule\(^1\), then we define strategy-proof and collusion-proof scoring profiles as the scores that each candidate receive in some strategy-proof and collusion-proof voting profiles respectively. For example, \((8, 2, 2)\) is a scoring profile in a plurality election with 12 voters and 3 candidates. This is also a 5-collusion-proof scoring profile since 5 new voters can not manipulate this election. The set of all possible scoring profiles in an election with \( n \) voters is denoted by \( S^n_r(C) \). When the voting rule \( r \) is clear from the context, we will drop \( r \) from the notation.

Clearly, since in score based voting rules, the winner is decided solely using the scores of the candidates, the above definitions are independent of the choice of strategy-proof or collusion-proof voting profiles if there are two or more voting profiles giving rise to the same scoring profile. The notion of collusion-proof scoring profiles has its motivation in computational social choice.

\[ 2.3 \text{ Societal Culture Assumption} \]

For scoring rules, the CM problem can also be thought of as if, we are given a scoring profile generated out of the votes of the truthful voters and the question remains the same. Here collusion-proofness of a scoring profile determines whether the hardness of the CM problem is at all required or not at that profile. The above notions naturally lead us to study a societal culture where all the scoring profiles are equally likely. We name this assumption the Impartial Scores Culture (ISC) assumption. The formal definition is as follows.

\textbf{Definition 2.6 (Impartial Scores Culture)}

Let \((s_1, \ldots, s_m)\) and \((s'_1, \ldots, s'_m)\) be two scoring profiles. Then the impartial scoring culture assumption states that,

\[ \text{Prob}(s_1, \ldots, s_m) = \text{Prob}(s'_1, \ldots, s'_m) \]

\(^1\)A voting rule is called a score based voting rule if winner is determined solely based on scores. Positional scoring rules are examples of such rules.
where $\text{Prob}(s_1, \ldots, s_m)$ and $\text{Prob}(s'_1, \ldots, s'_m)$ are the probabilities that the scoring profile of the society is $(s_1, \ldots, s_m)$ and $(s'_1, \ldots, s'_m)$ respectively.

### 2.4 Asymptotic Collusion-proof Voting Rule

With the above definitions, the notions of *asymptotic strategy-proofness* and *asymptotic collusion-proofness* on voters and candidates are defined as follows. Let $\mathcal{D}$ be a probability distribution over the voting profiles.

**Definition 2.7 (Asymptotic Strategy-proofness on Voters)** A voting rule $r$ is called asymptotically strategy-proof on voters if for all fixed and finite $C$,

$$\lim_{n \to \infty} \text{Prob}_{n \sim \mathcal{D}} \{ \succ^n \in T_r(C) \} = 1$$

In words, a voting rule is called asymptotically strategy-proof on voters if almost all the voting profiles are strategy-proof as we increase the number of voters. On similar lines, we define asymptotic strategy-proofness on candidates as follows.

**Definition 2.8 (Asymptotic Strategy-proofness on Candidates)** A voting rule $r$ is called asymptotically strategy-proof on candidates if $\exists N_0 \in \mathbb{N}$ such that $\forall n \geq N_0$,

$$\lim_{|C| \to \infty} \text{Prob}_{n \sim \mathcal{D}} \{ \succ^n \in T_r(C) \} = 1$$

The above concepts are generalized to asymptotic collusion-proofness as follows.

**Definition 2.9 (Asymptotic c-Collusion-proofness on Voters)** A voting rule $r$ is called asymptotically c-collusion-proof on voters if for all fixed and finite $C$,

$$\lim_{n \to \infty} \text{Prob}_{n \sim \mathcal{D}} \{ \succ^n \in T^c_r(C) \} = 1$$

**Definition 2.10 (Asymptotic c-Collusion-proofness on Candidates)** A voting rule $r$ is called asymptotically c-collusion-proof on candidates if $\exists N_0 \in \mathbb{N}$ such that $\forall n \geq N_0$,

$$\lim_{|C| \to \infty} \text{Prob}_{n \sim \mathcal{D}} \{ \succ^n \in T^c_r(C) \} = 1$$
Chapter 3

Results on asymptotic collusion-proofness

We show that with our new definition of strategy-proofness, many previously known results still hold. For almost all the voting rules studied in this work, we show that they are not asymptotically strategy-proof on candidates. That is, the problem of manipulation is really severe where computationally hard instances are. This shows that although there are evidences shown in the literature for computational hardness not being a very strong barrier, there is a good chance that the profiles which it is able to protect are really manipulable. We show the above results under both IC and ISC societal assumptions. IC assumption states that the voting profiles follow $U(L(C))^n$ probability distribution and the ISC assumes $U(S^n_r(C))$ probability distribution over scoring profiles. The following sections contain our results.

3.1 Plurality Voting Rule

We first provide a complete characterization of the scoring profiles which are strategy-proof under Plurality voting rule.

Proposition 3.1 For Plurality rule, following and only following scoring profiles $(x_1, x_2, \ldots, x_m) \in S^n([m])$ are strategy-proof:\footnote{Given a finite set $A$, $U(A)$ denotes the uniform probability distribution over $A$.}

1. (a) $|x_i - x_j| = 1$ or $0, \forall 1 \leq i, j \leq m$ with lexicographic tie breaking for the manipulator.

   (b) $x_1 = x_2 = \cdots = x_m$ with lexicographic tie breaking against the manipulator.

2. $\exists w \in [m]$ such that $x_w - x_i \geq 2, \forall 1 \leq i \leq m, i \neq w$, that is $c_w$ is the winner and its score is greater than that of every other candidate by at least 2.

\footnote{[$m] := \{1, 2, \ldots, m\}$.}
3. \( \exists w \in [m] \) such that \( x_w > x_i, \forall 1 \leq i \leq m, i \neq w \), with lexicographic tie breaking rule against the manipulator.

**Proof:** The first case talks about the situations where the score of all the candidates are as much same as possible. That is some candidates have highest score each and others have score just one less than highest. Now if the new voter’s most preferred candidate is already a candidate with highest score then her vote will make the candidate the winner. If her most preferred candidate has not scored highest then her vote will results in a tie and hence will make its candidate win only with lexicographic tie breaking rule for the manipulators. Hence the first case is strategy-proof.

The second and third cases deal with non-pivotal profiles, that is the new candidate cannot change current winner and hence she has no incentive to lie. Thus the scoring profile is strategy-proof.

In all other cases, the number of candidates tied for win or has score one less than the winner’s, is more than one and there is a candidate whose score is at least two less than the winner’s score. If the voter’s most preferred candidate is the candidate with least score then she cannot make her most preferred candidate win. But she can make one of the tied candidates and runner ups a winner depending upon the tie breaking rule being used. Hence these scoring profiles are manipulable.

\( \square \)

In the profiles in 1(a), the scores of the candidates are as same as possible, that is the scores of each pair of candidates differ by at most one. We denote the set of all these scoring profiles by \( E^u([m]) \) for any scoring rule. The set of all corresponding voting profiles are denoted by \( F^u([m]) \). For \( c \)-collusion-proofness, we have the following characterization.

**Proposition 3.2** For plurality rule, following and only following scoring profiles \( (x_1, x_2, \ldots, x_m) \in S^n([m]) \) are \( c \)-collusion (\( c>1 \)) proof:

\( \exists w \in \{1, 2, \ldots, m\} \) such that \( x_w - x_i \geq c + 1, \forall 1 \leq i \leq m, i \neq w \), that is \( c_w \) is the winner and its score is at least \( c+1 \) greater than that of every other candidate.

**Proof:** On similar line of the proof of the Proposition 3.1.

\( \square \)

The following lemma provides the cardinality of \( |S^n([m])| \) for plurality election.

**Lemma 3.1** For Plurality rule, \( |S^n([m])| = (n+m-1) \).

**Proof:** The set \( S^n([m]) \) is the solution set of the following equation.

\[
\sum_{i=1}^{m} x_i = n, x_i \in \{0,1,\ldots,n\}, \forall i \in [m]
\]
Hence $|S^n([m])| = \binom{n+m-1}{m-1}$. \hfill \Box

The theorem below bounds the fraction of $c-$collusion-proof scoring profiles for plurality voting rule.

**Theorem 3.1** Let $A$ be the set of all $c$-collusion-proof Plurality scoring profiles. Then $\frac{|A|}{|S^n([m])|} \geq \left(\frac{n-c}{n+1}\right)^{m-1}$.

**Proof:** Let $c$ be the coalition size; $c = o(n)$. Consider the following map,

$$f : S^{n-c-1}([m]) \rightarrow A,$$

$$x := (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_{cw} + c + 1, \ldots, x_m)$$

$c_{cw}$ is the winner of scoring profile $x$. $f$ is injective. Hence from Lemma 3.1,

$$|A| \geq |S^{n-c-1}([m])| = \binom{m+n-c-2}{m-1}$$

Now we have,

$$\frac{|A|}{|S^n([m])|} \geq \frac{(m+n-c-2)}{m-1} \left(\frac{m-n-1}{m-1}\right)^{m-1} = \frac{(m+n-c-2)\ldots(n-c)}{(m+n-1)\ldots(n+1)} \geq \left(\frac{n-c}{n+1}\right)^{m-1}$$

The last inequality comes from the fact that $\frac{a}{b} \geq \frac{a-1}{b-1}, \forall a, b \in \mathbb{N}, 0 < a < b$. \hfill \Box

The above bound immediately proves the following corollary for plurality rule.

**Corollary 3.1** Plurality rule under ISC assumption is asymptotically strategy-proof and asymptotically $o(n)$-collusion-proof on the number of voters.

**Proof:** Let $A$ be the set of all $c$-collusion-proof Plurality scoring profiles. let $c = o(n)$.

$$\lim_{n \to \infty} \frac{|A|}{|S^n([m])|} \geq \lim_{n \to \infty} \left(\frac{n-c}{n+1}\right)^{m-1} = \lim_{n \to \infty} \left(\frac{1-\frac{c}{n}}{1+\frac{1}{n}}\right)^{m-1} = 1$$
The last follows from the fact that \( c = o(n) \).

The above results imply that in elections where the number of voters, \( n \), is large, the problem of manipulation is not severe.

**Theorem 3.2** For Plurality rule, for \( m > n \), \( \frac{|E^n([m])|}{|S^n([m])|} \geq \left( \frac{m-n+1}{m} \right)^n \).

**Proof:** We have,

\[
\frac{|E^n([m])|}{|S^n([m])|} = \frac{\binom{m}{n}}{\binom{m+n-1}{n}} = \frac{m}{m+n-1} \cdot \frac{m-1}{m+n-2} \cdots \frac{m-n+1}{m} \geq \left( \frac{m-n+1}{m} \right)^n
\]

The last inequality comes from the fact that \( \frac{a}{b} \geq \frac{a-1}{b-1} \), \( \forall a, b \in \mathbb{N}, 0 < a < b \).

This theorem settles asymptotic collusion-proofness of plurality rule on the number of candidates as depicted in the corollary below.

**Corollary 3.2** Plurality rule under ISC assumption is asymptotically strategy-proof on candidates for lexicographic tie breaking rule for the manipulator, but not asymptotically strategy-proof on candidates with lexicographic tie breaking rule against manipulator. Plurality is not \( c \)-collusion-proof for \( c > 1 \) on candidates.

**Proof:** Fix some value for \( n \) - the number of voters.

\[
\lim_{m \to \infty} \frac{|E^n([m])|}{|S^n([m])|} \geq \lim_{m \to \infty} \left( \frac{m-n+1}{m} \right)^n = \lim_{m \to \infty} \left( 1 - \frac{n-1}{m} \right)^n = 1
\]

Now the results follow from the fact that the scoring profiles in \( E^n([m]) \) are strategy-proof on candidates for lexicographic tie breaking rule for the manipulator, but not asymptotically strategy-proof on candidates with lexicographic tie breaking rule against manipulator for plurality voting rule. Also the scoring profiles in \( E^n([m]) \) are not \( c \)-collusion-proof for \( c > 1 \) on candidates for plurality voting rule.
3.2 *k*-Approval Voting Rule

*k*-Approval voting rule is the scoring rule with the score vector \((1, \ldots, 1, 0, \ldots, 0)\), the first \(k\) entries are 1 and rest are 0. The scoring profiles are given by,

\[
S^n([m]) = \{(x_1, \ldots, x_m) \in \mathbb{N}^m : \sum_{i=1}^m x_i = nk, 0 \leq x_i \leq n, \forall i \in [m]\}
\]

\[|S^n([m])| = \text{Coefficient of } x^{nk} \text{ in } (1 + x + x^2 + \cdots + x^n)^m = \sum_{i=0}^{k} (-1)^i \binom{n(k-i)+m-1}{m-1}\]

**Proposition 3.3** For \(k\)-Approval rule with \(1 < k < m\), following and only following scoring profiles \((x_1, x_2, \ldots, x_m) \in S^n([m])\) are strategy-proof:

1. \(\exists w \in \{1, 2, \ldots, m\}\) such that \(x_w - x_i \geq 2, \forall 1 \leq i \leq m, i \neq w\), that is \(c_w\) is the winner and its score is at least two more than every other candidates.

2. \(x_1 = x_2 = \cdots = x_m\) with lexicographic tie breaking rule for the manipulator.

**Proof:** The first case deals with non-pivotal cases, i.e. the new candidate can not change current winner and hence he/she has no incentive to lie. Thus the scoring profile in non-manipulable. In the second case, the situation is same as with one voter and thus non-manipulable.

In all other cases, let \(W = \{c_1, \ldots, c_w\}\) be the set of all candidates with maximum score, \(R = \{c_{w+1}, \ldots, c_m\}\) be rest of the candidates. If the new voter’s true preference is \(c_1 > c_2 > \cdots c_w > c_{w+1} > \cdots c_m\) then new voter can clearly manipulate. \(\square\)

For \(k\)-Approval voting rule with \(1 < k < m\), we have the following results.

**Theorem 3.3** For \(k\)-Approval voting rule, \(\frac{|F^n([m])|}{|T^n([m])|} \geq \left(\frac{m-nk+1}{m}\right)^{nk}\).

**Proof:** We have,

\[
|F^n([m])| = |E^n([m])|(nk)!(m-k)!^n
= \binom{m}{nk}(nk)!(m-k)!^n
= m!((m-k)!)^n/(m-nk)!
\]

The first equality comes from the fact that, for each scoring profile in \(E^n([m])\), there are
(nk)!((m−k)!)^n many voting profiles which gives that scoring profile.

\[
\frac{|F^n([m])|}{|T^n([m])|} = \frac{m!}{(m-nk)!(m(m-1)\ldots(m-k+1))^n} \\
= \frac{m(m-1)\ldots(m-nk+1)}{(m(m-1)\ldots(m-k+1))^n} \\
\geq \left(\frac{m-nk+1}{m}\right)^nk
\]

The above theorem gives us the following corollaries.

**Corollary 3.3** Plurality rule under IC assumption is asymptotically strategy-proof on candidates for lexicographic tie breaking rule for the manipulator, but asymptotically manipulable on candidates with lexicographic tie breaking rule against manipulator. Plurality is not c-collusion-proof for c > 1 on candidates.

**Proof:** Plurality is 1−approval voting rule. Hence from Theorem 3.3, we get following.

\[
\lim_{n \to \infty} \frac{|F^n([m])|}{|T^n([m])|} \geq \lim_{n \to \infty} \left(\frac{m-n+1}{m}\right)^n \\
= \lim_{n \to \infty} \left(1 - \frac{n-1}{m}\right)^n \\
= 1
\]

The results follow from the fact that the voting profiles in \(F^n([m])\) are strategy-proof on candidates for lexicographic tie breaking rule for the manipulator, but not asymptotically strategy-proof on candidates with lexicographic tie breaking rule against manipulator for plurality voting rule. Also the voting profiles in \(F^n([m])\) are not c-collusion-proof for c > 1 on candidates for plurality voting rule.

**Corollary 3.4** For k > 1, k = O(1), k-Approval is not asymptotically strategy-proof on number of candidates under IC assumption.

**Proof:** For k > 1, k-Approval voting rule, the voting profiles in \(F^n([m])\) are strategy-proof.

\[
\lim_{m \to \infty} \frac{|F^n([m])|}{|T^n([m])|} \geq \lim_{m \to \infty} \left(\frac{m-nk+1}{m}\right)^{nk} \\
= \lim_{m \to \infty} \left(1 - \frac{nk-1}{m}\right)^{nk} \\
= 1
\]
The last equality follows from the fact that \( k = O(1) \).

The proposition below characterizes collusion-proof scoring profiles for \( k \)-approval voting rule.

**Proposition 3.4**  For \( k \)-Approval rule with \( 1 < k < m \), following scoring profiles \((x_1, x_2, \ldots, x_m) \in S^n([m])\) are \( c \)-collusion-proof:

1. \( \exists w \in \{1, 2, \ldots, m\} \) such that \( x_w - x_i \geq 2c, \forall 1 \leq i \leq m, i \neq w \), that is \( c_w \) is the winner and its score is at least two more than every other candidates.

**Proof:** The new \( c \) voters cannot change the outcome of the election and hence the profiles are \( c \)-collusion-proof. \( \square \)

**Theorem 3.4**  For \( 1 < k < m \), \( k \)-approval is asymptotically \( o(n) \)-collusion-proof on the number of voters under ISC assumption.

**Proof:** Define \( l := \binom{m-1}{k-1}, l' := \binom{m-2}{k-2} \). Let \( c \) be the coalition size. \( c = o(n) \). Let \( A \) be the set of all \( c \)-collusion-proof scoring profiles with \( n \) voters and \( m \) candidates. Consider the following map,

\[
f : S^{n-\ell c}([m]) \rightarrow A
\]

\[
(x_1, x_2, \ldots, x_m) \mapsto (x_1 + l', \ldots, x_w + l, \ldots, x_m + l')
\]

where \( c_w \) is the winner. Hence,

\[
|A| \geq |S^{n-\ell c}([m])| = \sum_{i=0}^{k} (-1)^i \binom{n-\ell c (k-i) + m - 1}{m - 1}
\]

Now,

\[
1 \geq \lim_{n \to \infty} \frac{|A|}{|S^n([m])|} \geq \lim_{n \to \infty} \frac{\sum_{i=0}^{k} (-1)^i \binom{n-\ell c (k-i) + m - 1}{m - 1}}{\sum_{i=0}^{k} (-1)^i \binom{n (k-i) + m - 1}{m - 1}} \geq 1
\]
The last equality comes from the fact that \( c = o(n) \), and \( l, l' \) are independent of \( n \). Hence the probability that an uniformly randomly chosen scoring profile is \( o(n) \)-collusion-proof goes to one as the number of voters increases and the result follows.

\[ \exists \frac{w \in \{1, 2, \ldots, m\}}{\text{such that } x_w - x_i \geq c + 1, \forall i} \]

Hence \( |S_n^{\text{veto}}([m])| = |S_n^{\text{plurality}}([m])| = \binom{n+m-1}{n} \). Theorem 3.4 implies that veto is asymptotically \( o(n) \)-collusion-proof on the number of voters under ISC assumption. The following proposition provides characterization for the collusion-proof scoring profiles under veto voting rule.

**Proposition 3.5** For veto rule following scoring profiles \((x_1, x_2, \ldots, x_m) \in S_n([m])\) are \(c\)-collusion-proof:

1. \( \exists w \in \{1, 2, \ldots, m\} \text{ such that } x_w - x_i \geq c + 1, \forall 1 \leq i \leq m \ i \neq w \), i.e. \( c_w \) is the winner and its score is at least two more than every other candidates.

2. For \( c < m - 1 \), \( x_1 = x_2 = \cdots = x_m \).

**Proof:** The first case deals with non-pivotal cases, i.e. the new voter can not change current winner and hence she has no incentive to lie. Thus the scoring profile in non-manipulable.

In the second case, the situation is same as with one voter and thus non-manipulable.

In all other cases, let \( W \) be the set of all candidates with maximum score, \( R \) be the set of all candidates with score one less than winners’ and \( L \) be the set of rest of the candidates. Not \( W \neq \emptyset \) since otherwise this will be same as case two. Hence at least one of \( R \) or \( L \) must be non-empty. Now if the least preferred candidate of new voters is not in \( W \) then new voters can certainly manipulate.

**Lemma 3.2** For Veto rule, for \( m > n \),

\[ \frac{|E_n^{\text{veto}}([m])|}{|S_n^{\text{veto}}([m])|} \geq \left( \frac{m-n+1}{m} \right)^n. \]

**Proof:** Consider the following map,

\[ f : E_n^{\text{veto}}([m]) \longrightarrow E_n^{\text{plurality}}([m]) \]

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\[(x_1, x_2, \ldots, x_m) \mapsto (-x_1, -x_2, \ldots, -x_m)\]

The map is clearly bijective and hence the result follows from Theorem 3.2. \(\square\)

Hence veto rule under ISC assumption is not asymptotically strategy-proof on candidates.

**Lemma 3.3** For Veto rule, for \(m > n\), 
\[
\frac{|F^n_v([m])|}{|S^n([m])|} \geq \left(\frac{m-n+1}{m}\right)^n.
\]

**Proof:** Follows from the fact that \(|F^n_v([m])| = |F^n_p([m])|\) and Theorem 3.3. \(\square\)

The above lemma shows that veto rule under IC assumption is not asymptotically strategy-proof on candidates. The following theorem bounds the number of \(c\)-collusion-proof strategy profiles for veto voting rule which will subsequently help us to prove \(o(n)\)-collusion-proofness of veto voting rule.

**Theorem 3.5** Let \(A\) be the set of all \(c\)-collusion-proof Veto scoring profiles. Then 
\[
\frac{|A|}{|S^n([m])|} \geq \left(\frac{n-(c+1)(m-1)+1}{mn-(c+1)(m-1)}\right)^{(c+1)(m-1)}.
\]

**Proof:** Let \(c\) be the coalition size. Consider the following map,
\[
f : S^{n-(c+1)(m-1)}([m]) \rightarrow A
\]
\[
x = (x_1, \ldots, x_m) \mapsto (x_1 - (c+1), \ldots, x_w, \ldots, x_m - (c+1))
\]

\(c_w\) is the winner of scoring profile \(x\). This map is clearly injective. Hence 
\[
|A| \geq |S^{n-(c+1)(m-1)}([m])| = \left(\frac{m+n-(c+1)(m-1)-1}{m-1}\right).
\]

We have,
\[
\frac{|A|}{|S^n([m])|} \geq \left(\frac{m+n-(c+1)(m-1)-1}{m-1}\right)^{(c+1)(m-1)}
\]

\[
= \frac{n \cdots (n-(c+1)(m-1)-1)}{(m+n-1) \cdots (m+n-(c+1)(m-1))}
\]

\[
\geq \left(\frac{n-(c+1)(m-1)+1}{mn-(c+1)(m-1)}\right)^{(c+1)(m-1)}
\]

The last inequality comes from the fact that \(\frac{a}{b} \geq \frac{a+1}{b+1}, \forall a, b \in \mathbb{N}, 0 < a < b.\) \(\square\)

So under ISC, Veto voting rule is asymptotically \(o(n)\)-collusion-proof on the number of voters.

### 3.4 Borda Voting Rule

For Borda voting rule, a score vector is \((m-1, m-2, \ldots, 0)\).
Proposition 3.6 For Borda rule, following and only following scoring profiles \((x_1, x_2, \ldots, x_m) \in S^n([m])\) are strategy-proof:

1. \(\exists w \in \{1, 2, \ldots, m\}\) such that \(x_w - x_i \geq m, \forall 1 \leq i \leq m, i \neq w\), that is \(c_w\) is the winner and its score is at least \(m\) greater than that of every other candidates.

2. \(x_1 = x_2 = \cdots = x_m\)

3. Every candidate except one is tied with highest score and the loser’s score is one less.

Proof: First case covers non-pivotal scoring profiles, i.e. current winner does not change irrespective of new voter’s vote. The second case is similar to one voter case. In the third case, it is always best response for a new voter to cast his/her true preference. Hence these are non-manipulable scoring profiles. Now we will show that every other scoring profiles are manipulable.

Case I - unique candidate with highest score: In this case, runner up candidate’s score is within \(m\) of winner’s score. Let \(c_w\) is the winner and \(c_r\) is a runner up. Consider a new voter with preference \(c_r \succ c_w \succ \cdots\). Now if the voter casts truthfully then \(c_w\) will be a winner. But casting \(c_r \succ \cdots \succ c_w\) makes \(c_r\) unique winner. Hence this scoring profile is manipulable.

Case II - more than one candidate with highest score: In this case more than one candidate has got highest score. Without loss of generality we may assume \(c_1, c_2, \ldots, c_i\) with \(1 < i < m\). If \(i < m - 1\) then a new voter with preference \(c_m \succ c_1 \succ c_2 \succ \cdots \succ c_i \succ \cdots\) is better off by casting \(c_m \succ \cdots \succ c_1 \succ c_2 \succ \cdots \succ c_i\). If \(i = m - 1\) then \(x_m < x_1 - 1\) and hence a voter with preference \(c_m \succ c_1 \succ c_2 \succ \cdots \succ c_{m-1}\) is better off by reporting \(c_1 \succ c_2 \succ \cdots \succ c_m\). Hence these profiles are manipulable. \(\square\)

Before going to the asymptotic result for the Borda voting rule, let us state some lemmas which we will be using later.

Lemma 3.4 \(\sum_{i=2}^{\infty} (-1)^i (i-1) \frac{x^i}{i!} = 1 - (1+x)e^{-x}, \forall x \in \mathbb{R}\).

Proof: We know,

\[
\begin{align*}
e^{-x} &= \sum_{i=0}^{\infty} (-1)^i \frac{x^i}{i!} \\
\Rightarrow -e^{-x} &= \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(i+1)x^i}{(i+1)!} \\
\Rightarrow -xe^{-x} &= \sum_{i=1}^{\infty} (-1)^i \frac{ix^i}{i!}
\end{align*}
\]
\[ -xe^{-x} = \sum_{i=2}^{\infty} \frac{(-1)^i (i-1) x^i}{i!} \]

\[ + \sum_{i=1}^{\infty} \frac{(-1)^i x^i}{i!} \]

\[ \Rightarrow -xe^{-x} = \sum_{i=2}^{\infty} \frac{(-1)^i (i-1) x^i}{i!} + e^{-x} - 1 \]

The second step follows by taking derivative with respect to \( x \) on both sides. The result follows from the last step.

Lemma 3.5 \( \forall l \in \mathbb{N} \), \( 1 - (-1)^l (l - 1) = \sum_{i=2}^{l-1} (-1)^i (i-1) \binom{l}{i} \).

**Proof:** We know that \( \forall l \in \mathbb{N}, x \in \mathbb{R}^+ \),

\[ (1 - x)^l = \sum_{i=0}^{l} (-1)^i \binom{l}{i} x^i \]

\[ \Rightarrow \frac{(1 - x)^l}{x} = \sum_{i=0}^{l} (-1)^i \binom{l}{i} x^{i-1} \]

\[ \Rightarrow -\frac{(1 - x)^l}{x^2} - l \frac{(1 - x)^{l-1}}{x} = \sum_{i=0}^{l} (i-1) (-1)^i \binom{l}{i} x^{i-2} \]

\[ \Rightarrow 1 - (-1)^l (l - 1) = \sum_{i=2}^{l-1} (-1)^i (i-1) \binom{l}{i} \]

The second step is derived by dividing both sides by \( x \). At third step, we take derivative with respect to \( x \). We have instantiated \( x \) to be 1 at fourth step.

The next lemma gives a useful cardinality formula. Suppose we have \( m \) finite sets namely \( A_1, \ldots, A_m \). Let \( A \) be the set of all elements which are present in at least two sets. Then the following lemma computes the cardinality of the set \( A \).

**Lemma 3.6** \( |A| = \sum_{r=2}^{m} (-1)^r (r-1) \sum_{I \in \binom{[m]}{r}} |A_I|^1 \).

**Proof:** Let us define \( |A| = \sum_{r=2}^{m} a_r \sum_{I \in \binom{[m]}{r}} |A_I| \) such that after \( r \) terms, all the elements which are present in at most \( r \) many sets have been counted exactly once. Hence it is enough to prove that,

\[ a_r = (-1)^r (r-1), \forall 2 \leq r \leq m. \]

\[ ^1\text{Let } A \text{ be a set. Then } \binom{[r]}{r} := \{ B \subseteq A : |B| = r \}. \text{ I be any subset of an index set. Then } A_I := \cap_{i \in I} A_i. \]
We will prove the above statement by induction on \( r \).

**Base case**: \( a_2 = 1 \) since no elements and particularly the elements present in exactly two sets have not been counted.

**Inductive step**: Let us assume the result for \( r \leq l - 1 \).

Now till the first \( l - 1 \) terms, the elements present in exactly \( l \) many sets have been counted exactly \( \sum_{i=2}^{l-1} a_i \binom{l}{i} \) many times. Hence,

\[
a_l = 1 - \sum_{i=2}^{l-1} a_i \binom{l}{i} = 1 - \sum_{i=2}^{l-1} (-1)^i (i-1) \binom{l}{i} = (-1)^l (l - 1)
\]

The second step follows from the inductive hypothesis and the third step follows from Lemma 3.5.

The following lemma gives a sufficient condition for a voting profile to be manipulable for large number of candidates.

**Lemma 3.7** For fixed \( n \) and large\(^1 \) \( m \), if in a voting profile \( (\succ_1, \ldots, \succ_n) \in \mathcal{L}(\mathcal{C})^n \), \( \exists c_i, c_j \in \mathcal{C}, c_i \neq c_j \) such that both \( c_i \) and \( c_j \) are ranked in the top \( l \) positions in each preference \( \succ_i, 1 \leq i \leq n \) where \((l - 1)n < m\), then the profile is manipulable.

**Proof**: Since we are working with voting rules and not with voting correspondences, the winner is unique. Hence WLOG WMA\(^2\) that \( c_i \) is not the winner. Let \( c_w \) be the winner. Then,

\[
\text{score}(c_w) \leq (m - 1)n \quad \text{and} \quad \text{score}(c_i) \geq (m - l)n
\]

Hence we have,

\[
\text{score}(c_w) - \text{score}(c_i) \leq (l - 1)n \leq m
\]

The above statement along with Proposition 3.6 proves the lemma.

**Theorem 3.6** Borda rule is not asymptotically strategy-proof on the number of candidates under IC assumption.

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\(^1\)large enough such that Item 2 and Item 3 of the Proposition 3.6 cannot occur.

\(^2\)WLOG : Without Loss Of Generality. WMA : We May Assume.
Proof: Consider an election with \( n \) voters, \( m \) candidates. Let us define \( l := \lambda m^{\frac{n-1}{n}}, \lambda \in \mathbb{N} \) where \( \lambda \) is a constant. Let us assume \( m \) to be large enough to satisfy the setting of Lemma 3.7. The existence of such an \( m \) is guaranteed from the fact that \( l = \Theta(m^{\frac{n-1}{n}}) \). Let us define \( A_i \) for \( i \in [m] \) as follows, 
\[ A_i := \{ \succ i \in \mathcal{L}(\mathcal{C})^n : i \text{ is ranked in the top } l \text{ positions in } \succ j, 1 \leq j \leq n \} \] 
Clearly, \( I \subseteq [m], |I| = k \) then,
\[ |A_I| = \begin{cases} ((m - l)!\Pi_{i=0}^{k-1} (l - i))^n, & \text{if } k \leq l \\ 0, & \text{if } k > l \end{cases} \]

Let us define the set \( A \) as follows,
\[ A := \{ \succ \in \mathcal{L}(\mathcal{C})^n : \exists i, j \in [m], i \neq j, \succ \in A_i, \succ \in A_j \} \]

Let us define the set of all manipulable voting profiles \( M^n([m]) \) as
\[ M^n([m]) := \mathcal{L}([m])^n \setminus T^n([m]) \]
From Lemma 3.7, it follows that \( A \subseteq M^n([m]) \). Now,
\[ \lim_{m \to \infty} \frac{|M^n([m])|}{|\mathcal{L}(\mathcal{C})^n|} \geq \frac{|A|}{(m!)^n} = \sum_{r=2}^l (-1)^r(r-1) \sum_{I \in \binom{[m]}{r}} \frac{|A_I|}{(m!)^n} = \sum_{r=2}^l (-1)^r(r-1) \binom{m}{r} \left( \Pi_{i=0}^{r-1} \frac{l - i}{m - i} \right)^n \sum_{r=2}^\infty (-1)^r(r-1) \frac{\lambda^r}{r!} = 1 - (1 + \lambda^n)e^{-\lambda^n} \]
The second step follows from Lemma 3.6, the fourth step is a result of using the fact that \( l = \Theta(m^{\frac{n-1}{n}}) \), and the last step follows from Lemma 3.4. Now since the above inequality is true for all \( \lambda \in \mathbb{R} \), we get,
\[ \lim_{m \to \infty} \frac{|M^n([m])|}{|\mathcal{L}(\mathcal{C})^n|} \geq \sup_{\lambda \in \mathbb{N}} \{ 1 - (1 + \lambda^n)e^{-\lambda^n} \} = 1 \]
Therefore the Borda rule is not asymptotically strategy-proof on the number of candidates under IC assumption.

The above theorem shows that it is highly likely that the voting profiles for which manipulating the Borda voting rule is intractable are actually manipulable. Hence computational hardness provides some resistance against manipulation of the Borda rule. The behavior of the Borda rule under ISC assumption is still open.
Chapter 4

Manipulation in Incomplete Information Setting

In a situation where researchers have found quite a few empirical and theoretical results showing weakness (if not non-existence) of complexity barrier against manipulation, we study a natural next step - manipulation in *incomplete information* setting. Historically the computational manipulation problem has been studied in *complete information* setting i.e. manipulators have complete information about the votes of non-manipulators. But this assumption is not valid for all but pathologically imaginary situations. Hence we study manipulability problem in the context of incomplete information. It turns out that to formulate manipulability in the incomplete information setting in the most natural way, ordinal setting is not enough; we need cardinal setting. Although voting theory has been studied classically in ordinal setting, there is some work [23] which justifies cardinal setting. [18, 12, 19] also justifies cardinal information setting specially in multi agent domain.

4.1 Definitions

In cardinal setting, preference of voter $v_i$ is given by $u^i = (u^i_1, u^i_2, \ldots, u^i_m)$ where $u^i$ satisfies some property $P(u^i)$. $P(u^i)$ can be for example $u^i_j \in [0, 1]$ or $\sum_j u^i_j = 1$ or more generally $u^i \in X$, where $X \subset \mathbb{R}^n$.

**Definition 4.1** (*DSIC voting rule*) The voting rule is said to be dominant strategy incentive compatible (*DSIC*) if $u^i(r(u)) \geq u^i(r(u')) \quad \forall i \in \{1, \ldots, n\} \quad \forall u, u' \in U^n$, $u$ and $u'$ differs in $i^{th}$ coordinate only, $\forall n \in \mathbb{N}^+$

A voting rule is called manipulable if it is not incentive compatible.
Let $\delta$ be a probability distribution over $\mathcal{C}$, i.e. $\delta \in \Delta(\mathcal{C})$, then we define $u(\delta) := \sum_{c_i \in \mathcal{C}} \delta(c_i).u(c_i)$.

Let $\mathcal{G}_m$ is the set of all permutation of $\{1, 2, \ldots, m\}$, $\mathcal{G}_m^n := \times_{i=1}^n \mathcal{G}_m$ and $\delta_m^n \in \Delta(\mathcal{G}_m^n)$ and $r$ be a voting rule, then we can define winning probability of a candidate $c_i \in \mathcal{C}$ as follows:

$$p_{c_i}^{(r)}(\delta_m^n, x) := \sum_{(s_1, \ldots, s_n) \in \mathcal{G}_m^n \atop c_i = r(s_1, \ldots, s_n)} \delta_m^n(s_1, \ldots, s_n), \forall x \in \mathcal{G}_m.$$  

We assume that each voter has a utility function $u : \mathcal{C} \rightarrow [0, 1]$. $u(c_i)$ is the utility that the voter gets if the candidate $c_i$ wins. We will denote the set of all utility functions $u : \mathcal{C} \rightarrow [0, 1]$ by $[0, 1]^\mathcal{C}$. Now we have a natural definition of non-manipulability.

**Definition 4.2** $\delta_m^n$ is $\epsilon$-non-manipulable under a voting rule $r$ if

$$|u(p_{c_i}^{(r)}(\delta_m^n, x)) - u(p_{c_i}^{(r)}(\delta_m^n, x'))| \leq \epsilon,$$

$$\forall x, x' \in \mathcal{G}_m, \forall u \in [0, 1]^\mathcal{C}$$

That is,

$$|\sum_{c_i \in \mathcal{C}} (p_{c_i}^{(r)}(\delta_m^n, x) - p_{c_i}^{(r)}(\delta_m^n, x'))u(c_i)| \leq \epsilon,$$

$$\forall x, x' \in \mathcal{G}_m, \forall u \in [0, 1]^\mathcal{C}$$

### 4.2 Results

With this definition, we prove following interesting results in cardinal incomplete information settings.

**Lemma 4.1** Let $\delta_m^n \in \mathcal{G}_m^n$ and $r$ be a voting rule, If $\forall x, x' \in \mathcal{G}_m$, $\sum_{c_i \in \mathcal{C}} |p_{c_i}^{(r)}(\delta_m^n, x) - p_{c_i}^{(r)}(\delta_m^n, x')| \leq \epsilon$ then $\delta_m^n$ is $\epsilon$-non-manipulable under voting rule $r$.

**Proof:** Directly follows from definition 4.2

**Lemma 4.2** Let $\delta_m^n \in \mathcal{G}_m^n$, if the total probability mass on pivotal profiles (in the sense of complete information setting) is less than $\frac{\epsilon}{2m}$ i.e.

$$\sum_{(s_1, \ldots, s_n) \in \mathcal{G}_m^n \atop (s_1, \ldots, s_n) \in \text{PVP}_{m,n}} \delta_m^n(s_1, \ldots, s_n) \leq \frac{\epsilon}{2m},$$
Then $\delta^n_m$ is $\epsilon$-non-manipulable under voting rule $r$.

Proof:

$$p_{ci}^{(r)}(\delta^n_m, x) := \sum_{(s_1, \ldots, s_n) \in S^n_m, c_i = r(s_1, \ldots, s_n, x)} \delta^n_m(s_1, \ldots, s_n)$$

$$= \sum_{(s_1, \ldots, s_n) \in S^n_m, c_i = r(s_1, \ldots, s_n, x)} \delta^n_m(s_1, \ldots, s_n) + \sum_{(s_1, \ldots, s_n) \in PV F_{m,n}^{(r)}, c_i = r(s_1, \ldots, s_n, x')} \delta^n_m(s_1, \ldots, s_n)$$

$$= \sum_{(s_1, \ldots, s_n) \in S^n_m, c_i = r(s_1, \ldots, s_n, x)} \delta^n_m(s_1, \ldots, s_n) + \sum_{(s_1, \ldots, s_n) \in PV F_{m,n}^{(r)}, c_i = r(s_1, \ldots, s_n, x')} \delta^n_m(s_1, \ldots, s_n)$$

Therefore,

$$\sum_{c_i \in E} |p_{ci}^{(r)}(\delta^n_m, x) - p_{ci}^{(r)}(\delta^n_m, x')|$$

$$= \sum_{c_i \in E} \left| \sum_{(s_1, \ldots, s_n) \in S^n_m, c_i = r(s_1, \ldots, s_n, x)} \delta^n_m(s_1, \ldots, s_n) - \sum_{(s_1, \ldots, s_n) \in S^n_m, c_i = r(s_1, \ldots, s_n, x')} \delta^n_m(s_1, \ldots, s_n) \right|$$

$$\leq \sum_{c_i \in E} \left| \sum_{(s_1, \ldots, s_n) \in S^n_m, c_i = r(s_1, \ldots, s_n, x)} \delta^n_m(s_1, \ldots, s_n) \right| + \sum_{c_i \in E} \left| \sum_{(s_1, \ldots, s_n) \in S^n_m, c_i = r(s_1, \ldots, s_n, x')} \delta^n_m(s_1, \ldots, s_n) \right|$$

$$= m \frac{\epsilon}{2m} + m \frac{\epsilon}{2m}$$

$$= \epsilon$$

Lemma 4.3 Let $\delta^n_m \in S^n_m$, if $\delta^n_m$ is $\frac{\epsilon}{m}$-pivotal then $\delta^n_m$ is $\epsilon$-non-manipulable under voting rule $r$.

Now we would like to find the conditions exactly when a voting profile is $\epsilon$-non-manipulable.

$\delta^n_m \in S^n_m$ and $r$ be a voting rule, then $\delta^n_m$ is $\epsilon$-non-manipulable if by definition of non-manipulability

$$|u((p_{ci}^{(r)}(\delta^n_m, x))_{c_i \in E}) - u((p_{ci}^{(r)}(\delta^n_m, x'))_{c_i \in E})| \leq \epsilon, \forall x, x' \in S_m, \forall u \in [0, 1]^E$$
\[ \Leftrightarrow \sum_{c_i \in \mathcal{C}} (p^{(r)}(\delta^n_m, x) - p^{(r)}(\delta^n_m, x')) u(c_i) \leq \epsilon, \forall x, x' \in \mathcal{S}_m, \forall u \in [0, 1]^e \]

\[ \Leftrightarrow \sum_{c_i \in \mathcal{C}} (p^{(r)}(\delta^n_m, x) - p^{(r)}(\delta^n_m, x')) \leq \epsilon \]

and \[ \sum_{c_i \in \mathcal{C}} (p^{(r)}(\delta^n_m, x') - p^{(r)}(\delta^n_m, x)) \leq \epsilon \forall x, x' \in \mathcal{S}_m, \forall u \in [0, 1]^e \]

We can write this as a theorem as it will have many important corollaries.

**Theorem 4.1** \( \delta^n_m \in \mathcal{S}_m^n \) and \( r \) be a voting rule, then \( \delta^n_m \) is a \( \epsilon \)-non-manipulable iff

\[ \sum_{c_i \in \mathcal{C}} (p^{(r)}(\delta^n_m, x) - p^{(r)}(\delta^n_m, x')) \leq \epsilon \]

and \[ \sum_{c_i \in \mathcal{C}} (p^{(r)}(\delta^n_m, x') - p^{(r)}(\delta^n_m, x)) \leq \epsilon \]

\forall x, x' \in \mathcal{S}_m, \forall u \in [0, 1]^e

We have the following immediate corollaries,

**Corollary 4.1** \( \delta^n_m \in \mathcal{S}_m^n \) and \( r \) be a voting rule, if \( \exists x, x' \in \mathcal{S}_m \) such that \( \sum_{c_i \in \mathcal{C}} |p^{(r)}(\delta^n_m, x) - p^{(r)}(\delta^n_m, x')| > 2\epsilon \) then \( \delta^n_m \in \mathcal{S}_m^n \) is \( \epsilon \)-manipulable under voting rule \( r \).

**Corollary 4.2** \( \delta^n_m \in \mathcal{S}_m^n \) and \( r \) be a voting rule, if \( \max_{x, x' \in \mathcal{S}_m} \{p^{(r)}(\delta^n_m, x) - p^{(r)}(\delta^n_m, x')\} \leq \frac{\epsilon}{m} \) then \( \delta^n_m \) is \( \epsilon \)-non-manipulable.
Chapter 5

Conclusion and Future Work

For all the voting rules studied here, the severity of manipulation has been shown to be present mostly in the elections with large number of candidates and not in the elections with many voters. Also this is the first work, to the best of our knowledge, which tries to address the reverse question posed in the beginning - are the profiles which are being protected by computational hardness at all manipulable or not? We have studied some popular voting rules in this work. Certainly a general result would be much more interesting which is a prospective future work. More specifically, the techniques used in the proofs of the theorems in this thesis work does not apply to voting rules other than scoring rules. Hence studying asymptotic behavior of voting rules other than scoring rules seems to be a challenging future working direction.

We have explored a new societal cultural assumption called ISC and argued for its need in the context of computational social choice. We notice a positive correlation among results under ISC assumption and under IC assumption. Again a general result connecting these two concepts is another important future direction of research. Here we conjecture that a voting procedure is asymptotically strategy-proof or $c$-collusion proof either on the number of voters or on the number of candidates under ISC assumption if and only if it is so under IC assumption. We also mention here that a similar correlation can be seen between IC and IAC culture assumptions although there is no such result connecting these two societies. An interesting future direction is to find axiomatic characterization of asymptotically collusion-proof voting rules.

For manipulability in the incomplete information setting, our results provide insight into manipulation in many realistic situations. However to study CM problem in this setting, we need to formulate a related computational problem. The primary hurdle seems to have a succinct representation for probability distribution. Although some special distributions can be specified neatly using parameters, a structure to represent a probability distribution in its full generality is yet to be found.
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